

EMBEDDINGS BETWEEN WEIGHTED COMPLEMENTARY LOCAL MORREY-TYPE SPACES AND WEIGHTED LOCAL MORREY-TYPE SPACES

AMIRAN GOGATISHVILI, RZA MUSTAFAYEV, AND TUĞÇE ÜNVER

ABSTRACT. In this paper embeddings between weighted complementary local Morrey-type spaces ${}^cLM_{p\theta,\omega}(\mathbb{R}^n, v)$ and weighted local Morrey-type spaces $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ are characterized. In particular, two-sided estimates of the optimal constant c in the inequality

$$\left(\int_0^\infty \left(\int_{B(0,t)} f(x)^{p_2} v_2(x) dx \right)^{\frac{q_2}{p_2}} u_2(t) dt \right)^{\frac{1}{q_2}} \leq c \left(\int_0^\infty \left(\int_{{}^cB(0,t)} f(x)^{p_1} v_1(x) dx \right)^{\frac{q_1}{p_1}} u_1(t) dt \right)^{\frac{1}{q_1}}$$

are obtained, where $p_1, p_2, q_1, q_2 \in (0, \infty)$, $p_2 \leq q_2$ and u_1, u_2 and v_1, v_2 are weights on $(0, \infty)$ and \mathbb{R}^n , respectively. The proof is based on the combination of duality techniques with estimates of optimal constants of the embeddings between weighted local Morrey-type and complementary local Morrey-type spaces and weighted Lebesgue spaces, which reduce the problem to the solutions of the iterated Hardy-type inequalities.

1. INTRODUCTION

The classical Morrey spaces $\mathcal{M}_{p,\lambda} \equiv \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$, were introduced by C. Morrey in [18] in order to study regularity questions which appear in the Calculus of Variations, and defined as follows: for $0 \leq \lambda \leq n$ and $1 \leq p \leq \infty$,

$$\mathcal{M}_{p,\lambda} := \left\{ f \in L_p^{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{x \in \mathbb{R}^n, r > 0} r^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $B(x, r)$ is the open ball centered at x of radius r .

Note that $\mathcal{M}_{p,0}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$ and $\mathcal{M}_{p,n}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

These spaces describe local regularity more precisely than Lebesgue spaces and appeared to be quite useful in the study of the local behavior of solutions to partial differential equations, a priori estimates and other topics in PDE (cf. [13]).

The classical Morrey spaces were widely investigated during the last decades, including the study of classical operators of Harmonic and Real Analysis - maximal, singular and potential operators - in generalizations of these spaces (so-called Morrey-type spaces). The local Morrey-type spaces and the complementary local Morrey-type spaces introduced by Guliyev in his doctoral thesis [16].

The local Morrey-type spaces $LM_{p\theta,\omega}$ and the complementary local Morrey-type spaces ${}^cLM_{p\theta,\omega}$ were intensively studied during the last decades. The research mainly includes the study of the boundedness of classical operators in these spaces (see, for instance, [2–10]), and the investigation of the functional-analytic properties of them and relation of these spaces with other known function spaces (see, for instance, [1, 11, 19]). We refer the reader to the surveys [2] and [3] for a comprehensive discussion of the history of $LM_{p\theta,\omega}$ and ${}^cLM_{p\theta,\omega}$.

Let A be any measurable subset of \mathbb{R}^n , $n \geq 1$. By $\mathfrak{M}(A)$ we denote the set of all measurable functions on A . The symbol $\mathfrak{M}^+(A)$ stands for the collection of all $f \in \mathfrak{M}(A)$ which are non-negative on A . The family of all weight functions (also called just weights) on A , that is, measurable, positive and finite a.e. on A , is given by $\mathcal{W}(A)$.

For $p \in (0, \infty]$, we define the functional $\|\cdot\|_{p,A}$ on $\mathfrak{M}(A)$ by

$$\|f\|_{p,A} := \begin{cases} \left(\int_A |f(x)|^p dx \right)^{1/p} & \text{if } p < \infty \\ \text{ess sup}_A |f(x)| & \text{if } p = \infty \end{cases}.$$

If $w \in \mathcal{W}(A)$, then the weighted Lebesgue space $L_p(w, A)$ is given by

$$L_p(w, A) \equiv L_{p,w}(A) := \{f \in \mathfrak{M}(A) : \|f\|_{p,w,A} := \|fw\|_{p,A} < \infty\}.$$

When $A = \mathbb{R}^n$, we often write simply $L_{p,w}$ and $L_p(w)$ instead of $L_{p,w}(A)$ and $L_p(w, A)$, respectively.

2010 *Mathematics Subject Classification.* Primary 46E30; Secondary 26D10.

Key words and phrases. local Morrey-type spaces, embeddings, iterated Hardy inequalities.

Throughout the paper, we always denote by c and C a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$, ($b \gtrsim a$) we mean that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. We will denote by $\mathbf{1}$ the function $\mathbf{1}(x) = 1$, $x \in \mathbb{R}$.

Given two quasi-normed vector spaces X and Y , we write $X = Y$ if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol $X \hookrightarrow Y$ ($Y \hookleftarrow X$) means that $X \subset Y$ and the natural embedding I of X in Y is continuous, that is, there exist a constant $c > 0$ such that $\|z\|_Y \leq c\|z\|_X$ for all $z \in X$. The best constant of the embedding $X \hookrightarrow Y$ is $\|I\|_{X \rightarrow Y}$.

The weighted local Morrey-type spaces $LM_{p\theta,\omega}(\mathbb{R}^n, \nu)$ and weighted complementary local Morrey-type spaces ${}^cLM_{p\theta,\omega}(\mathbb{R}^n, \nu)$ are defined as follows: Let $0 < p, \theta \leq \infty$. Assume that $\omega \in \mathfrak{M}^+(0, \infty)$ and $\nu \in \mathcal{W}(\mathbb{R}^n)$.

$$LM_{p\theta,\omega}(\mathbb{R}^n, \nu) := \left\{ f \in L_{p,\nu}^{\text{loc}}(\mathbb{R}^n) : \|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n, \nu)} < \infty \right\},$$

where

$$\|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n, \nu)} := \left\| \|f\|_{p,\nu,B(0,r)} \right\|_{\theta,\omega,(0,\infty)},$$

and

$${}^cLM_{p\theta,\omega}(\mathbb{R}^n, \nu) := \left\{ f \in \bigcap_{t>0} L_{p,\nu}({}^cB(0,t)) : \|f\|_{{}^cLM_{p\theta,\omega}(\mathbb{R}^n, \nu)} < \infty \right\},$$

where

$$\|f\|_{{}^cLM_{p\theta,\omega}(\mathbb{R}^n, \nu)} := \left\| \|f\|_{p,\nu,{}^cB(0,r)} \right\|_{\theta,\omega,(0,\infty)}.$$

Remark 1.1. In [5] and [7] it were proved that the spaces $LM_{p\theta,\omega}(\mathbb{R}^n) := LM_{p\theta,\omega}(\mathbb{R}^n, \mathbf{1})$ and ${}^cLM_{p\theta,\omega}(\mathbb{R}^n) := {}^cLM_{p\theta,\omega}(\mathbb{R}^n, \mathbf{1})$ are non-trivial, i.e. consists not only of functions equivalent to 0 on \mathbb{R}^n , if and only if

$$(1.1) \quad \|\omega\|_{\theta,(t,\infty)} < \infty, \quad \text{for some } t > 0,$$

and

$$(1.2) \quad \|\omega\|_{\theta,(0,t)} < \infty, \quad \text{for some } t > 0,$$

respectively. The same conclusion is true for $LM_{p\theta,\omega}(\mathbb{R}^n, \nu)$ and ${}^cLM_{p\theta,\omega}(\mathbb{R}^n, \nu)$ for any $\nu \in \mathcal{W}(\mathbb{R}^n)$.

The proof of the following statement is straightforward.

Lemma 1.2. (i) If $\|\omega\|_{\theta,(t_1,\infty)} = \infty$ for some $t_1 > 0$, then

$$f \in LM_{p\theta,\omega}(\mathbb{R}^n, \nu) \Rightarrow f = 0 \quad \text{a.e. on } B(0, t_1).$$

(ii) If $\|\omega\|_{\theta,(0,t_2)} = \infty$ for some $t_2 > 0$, then

$$f \in {}^cLM_{p\theta,\omega}(\mathbb{R}^n, \nu) \Rightarrow f = 0 \quad \text{a.e. on } {}^cB(0, t_2).$$

Let $0 < \theta \leq \infty$. We denote by

$$\Omega_\theta := \{\omega \in \mathfrak{M}^+(0, \infty) : 0 < \|\omega\|_{\theta,(t,\infty)} < \infty, t > 0\},$$

$${}^c\Omega_\theta := \{\omega \in \mathfrak{M}^+(0, \infty) : 0 < \|\omega\|_{\theta,(0,t)} < \infty, t > 0\}.$$

Let $\nu \in \mathcal{W}(\mathbb{R}^n)$. It is easy to see that $LM_{p\theta,\omega}(\mathbb{R}^n, \nu)$ and ${}^cLM_{p\theta,\omega}(\mathbb{R}^n, \nu)$ are quasi-normed vector spaces when $\omega \in \Omega_\theta$ and $\omega \in {}^c\Omega_\theta$, respectively.

The following statements are immediate consequences of Fubini's Theorem and were observed in [5] and [7], for $\nu = 1$, respectively.

Lemma 1.3. Let $0 < p \leq \infty$ and $\nu \in \mathcal{W}(\mathbb{R}^n)$. Then

(i) $LM_{pp,\omega}(\mathbb{R}^n, \nu) = L_p(w)$, where $w(x) := \nu(x)\|\omega\|_{p,([x],\infty)}$, $x \in \mathbb{R}^n$.

(ii) ${}^cLM_{pp,\omega}(\mathbb{R}^n, \nu) = L_p(w)$, where $w(x) := \nu(x)\|\omega\|_{p,(0,[x])}$, $x \in \mathbb{R}^n$.

Recall that the embeddings between weighted local Morrey-type spaces and weighted Lebesgue spaces, that is, the embeddings

$$(1.3) \quad L_{p_1}(\nu_1) \hookrightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, \nu_2),$$

$$(1.4) \quad L_{p_1}(\nu_1) \hookrightarrow {}^cLM_{p_2\theta,\omega}(\mathbb{R}^n, \nu_2),$$

$$(1.5) \quad L_{p_1}(v_1) \hookleftarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2),$$

$$(1.6) \quad L_{p_1}(v_1) \hookleftarrow {}^cLM_{p_2\theta,\omega}(\mathbb{R}^n, v_2)$$

are completely characterized in [19].

Our principal goal in this paper is to investigate the embeddings between weighted complementary local Morrey-type spaces and weighted local Morrey type spaces and vice versa, that is, the embeddings

$$(1.7) \quad {}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2),$$

$$(1.8) \quad LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \hookrightarrow {}^cLM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2).$$

An approach used in this paper consist of a duality argument combined with estimates of optimal constants of embeddings (1.3) - (1.6), which reduce the problem to the solutions of the iterated Hardy-type inequalities

$$(1.9) \quad \left\| \|H^* f\|_{p,u,(0,\cdot)} \right\|_{q,w,(0,\infty)} \leq c \|f\|_{\theta,v,(0,\infty)}, \quad f \in \mathfrak{M}^+(0, \infty),$$

with

$$(H^* f)(t) := \int_t^\infty f(\tau) d\tau, \quad t > 0,$$

where u, v, w are weights on $(0, \infty)$ and $0 < p, q \leq \infty, 1 < \theta < \infty$. There exists different solutions of these inequalities. We will use characterizations from [14] and [15].

Note that in view of Lemma 1.3, embeddings (1.7) - (1.8) contain embeddings (1.3) - (1.6) as a special case. Moreover, by the change of variables $x = y/|y|^2$ and $t = 1/\tau$, it is easy to see that (1.8) is equivalent to the embedding

$${}^cLM_{p_1\theta_1,\tilde{\omega}_1}(\mathbb{R}^n, \tilde{v}_1) \hookrightarrow LM_{p_2\theta_2,\tilde{\omega}_2}(\mathbb{R}^n, \tilde{v}_2),$$

where $\tilde{v}_i(y) = v_i(y/|y|^2)|y|^{-2n/p_i}$ and $\tilde{\omega}_i(\tau) = \tau^{-2/\theta_i}\omega_i(1/\tau)$, $i = 1, 2$. This note allows us to concentrate our attention on characterization of (1.7). On the negative side of things we have to admit that the duality approach works only in the case when, in (1.7) - (1.8), one has $p_2 \leq \theta_2$. Unfortunately, in the case when $p_2 > \theta_2$ the characterization of these embeddings remains open.

In particular, we obtain two-sided estimates of the optimal constant c in the inequality

$$\left(\int_0^\infty \left(\int_{B(0,t)} f(x)^{p_2} v_2(x) dx \right)^{\frac{q_2}{p_2}} u_2(t) dt \right)^{\frac{1}{q_2}} \leq c \left(\int_0^\infty \left(\int_{B(0,t)} f(x)^{p_1} v_1(x) dx \right)^{\frac{q_1}{p_1}} u_1(t) dt \right)^{\frac{1}{q_1}},$$

where $p_1, p_2, q_1, q_2 \in (0, \infty)$, $p_2 \leq q_2$ and u_1, u_2 and v_1, v_2 are weights on $(0, \infty)$ and \mathbb{R}^n , respectively.

The paper is organized as follows. We start with formulations of our main results in Section 2. The proofs of the main results are presented in Section 3.

2. STATEMENT OF THE MAIN RESULTS

We adopt the following usual conventions.

Convention 2.1. (i) Throughout the paper we put $0/0 = 0$, $0 \cdot (\pm\infty) = 0$ and $1/(\pm\infty) = 0$.

(ii) We put

$$p' := \begin{cases} \frac{p}{1-p} & \text{if } 0 < p < 1, \\ \infty & \text{if } p = 1, \\ \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ 1 & \text{if } p = \infty. \end{cases}$$

(iii) To state our results we use the notation $p \rightarrow q$ for $0 < p, q \leq \infty$ defined by

$$\frac{1}{p \rightarrow q} = \frac{1}{q} - \frac{1}{p} \quad \text{if } q < p,$$

and $p \rightarrow q = \infty$ if $q \geq p$.

(iv) If $I = (a, b) \subseteq \mathbb{R}$ and g is a monotone function on I , then by $g(a)$ and $g(b)$ we mean the limits $\lim_{t \rightarrow a+} g(t)$ and $\lim_{t \rightarrow b-} g(t)$, respectively.

Our main results are the following theorems. Throughout the paper we will denote

$$\widetilde{V}(x) := \|v_1^{-1}v_2\|_{p_1 \rightarrow p_2, B(0, x)}, \quad \text{and} \quad \mathcal{V}(t, x) := \frac{\widetilde{V}(t)}{\widetilde{V}(t) + \widetilde{V}(x)} \quad (t > 0, x > 0).$$

Theorem 2.2. Let $0 < \theta_2 = p_2 \leq p_1 = \theta_1 < \infty$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Then

$$\|I\|_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \left\| \|\omega_1\|_{p_1,(0,|\cdot|)}^{-1} \|\omega_2\|_{p_2,(\cdot,|\cdot|,\infty)} \right\|_{p_1 \rightarrow p_2, v_1^{-1}v_2, \mathbb{R}^n}.$$

Theorem 2.3. Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$ and $\theta_2 \neq p_2 \leq p_1 = \theta_1$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$.

(i) If $p_1 \leq \theta_2$, then

$$\|I\|_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t \in (0, \infty)} \left\| \|\omega_1\|_{p_1,(0,|\cdot|)}^{-1} \right\|_{p_1 \rightarrow p_2, v_1^{-1}v_2, B(0,t)} \|\omega_2\|_{\theta_2,(t,\infty)};$$

(ii) If $\theta_2 < p_1$, then

$$\|I\|_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \left(\int_0^\infty \left\| \|\omega_1\|_{p_1,(0,|\cdot|)}^{-1} \right\|_{p_1 \rightarrow p_2, v_1^{-1}v_2, B(0,t)}^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \rightarrow \theta_2} \right) \right)^{\frac{1}{p_1 \rightarrow \theta_2}}.$$

Theorem 2.4. Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$ and $\theta_2 = p_2 \leq p_1 \neq \theta_1$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$.

(i) If $\theta_1 \leq p_2$, then

$$\|I\|_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t \in (0, \infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-1} \left\| \|\omega_2\|_{p_2,(\cdot,|\cdot|,\infty)} \right\|_{p_1 \rightarrow p_2, v_1^{-1}v_2, B(0,t)};$$

(ii) If $p_2 < \theta_1$, then

$$\begin{aligned} \|I\|_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} &\approx \left(\int_0^\infty \left\| \|\omega_2\|_{p_2,(\cdot,|\cdot|,\infty)} \right\|_{p_1 \rightarrow p_2, v_1^{-1}v_2, B(0,t)}^{\theta_1 \rightarrow p_2} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p_2} \right) \right)^{\frac{1}{\theta_1 \rightarrow p_2}} \\ &\quad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\| \|\omega_2\|_{p_2,(\cdot,|\cdot|,\infty)} \right\|_{p_1 \rightarrow p_2, v_1^{-1}v_2, \mathbb{R}^n}. \end{aligned}$$

In view of Lemma 1.3, Theorems 2.2 - 2.4 are straightforward consequences of [19, Theorem 3.1] and [19, Theorem 4.2].

To state further results we need the following definitions.

Definition 2.5. Let U be a continuous, strictly increasing function on $[0, \infty)$ such that $U(0) = 0$ and $\lim_{t \rightarrow \infty} U(t) = \infty$. Then we say that U is admissible.

Let U be an admissible function. We say that a function φ is U -quasiconcave if φ is equivalent to an increasing function on $(0, \infty)$ and φ/U is equivalent to a decreasing function on $(0, \infty)$. We say that a U -quasiconcave function φ is non-degenerate if

$$\lim_{t \rightarrow 0+} \varphi(t) = \lim_{t \rightarrow \infty} \frac{1}{\varphi(t)} = \lim_{t \rightarrow \infty} \frac{\varphi(t)}{U(t)} = \lim_{t \rightarrow 0+} \frac{U(t)}{\varphi(t)} = 0.$$

The family of non-degenerate U -quasiconcave functions is denoted by \mathcal{Q}_U .

Definition 2.6. Let U be an admissible function, and let w be a non-negative measurable function on $(0, \infty)$. We say that the function φ , defined by

$$\varphi(t) = U(t) \int_0^\infty \frac{w(\tau) d\tau}{U(\tau) + U(t)}, \quad t \in (0, \infty),$$

is a fundamental function of w with respect to U . One will also say that $w(\tau) d\tau$ is a representation measure of φ with respect to U .

Remark 2.7. Let φ be the fundamental function of w with respect to U . Assume that

$$\int_0^\infty \frac{w(\tau) d\tau}{U(\tau) + U(t)} < \infty, \quad t > 0, \quad \int_0^1 \frac{w(\tau) d\tau}{U(\tau)} = \int_1^\infty w(\tau) d\tau = \infty.$$

Then $\varphi \in \mathcal{Q}_U$.

Remark 2.8. Suppose that $\varphi(x) < \infty$ for all $x \in (0, \infty)$, where φ is defined by

$$\varphi(x) = \operatorname{ess\,sup}_{t \in (0, x)} U(t) \operatorname{ess\,sup}_{\tau \in (t, \infty)} \frac{w(\tau)}{U(\tau)}, \quad t \in (0, \infty).$$

If

$$\limsup_{t \rightarrow 0+} w(t) = \limsup_{t \rightarrow +\infty} \frac{1}{w(t)} = \limsup_{t \rightarrow 0+} \frac{U(t)}{w(t)} = \limsup_{t \rightarrow +\infty} \frac{w(t)}{U(t)} = 0,$$

then $\varphi \in Q_U$.

Theorem 2.9. Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$, $p_2 < p_1$, $\theta_1 \leq p_2 < \theta_2$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Suppose that \tilde{V} is admissible and

$$\varphi_1(x) := \sup_{t \in (0, \infty)} \tilde{V}(t) \mathcal{V}(x, t) \|\omega_1\|_{\theta_1, (0, t)}^{-1} \in Q_{\tilde{V}^{-\frac{1}{p_1 - p_2}}}.$$

(i) If $p_1 \leq \theta_2$, then

$$\|I\|_{{}^cLM_{p_1\theta_1, \omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2, \omega_2}(\mathbb{R}^n, v_2)} \approx \sup_{x \in (0, \infty)} \varphi_1(x) \sup_{t \in (0, \infty)} \mathcal{V}(t, x) \|\omega_2\|_{\theta_2, (t, \infty)}.$$

(ii) If $\theta_2 < p_1$, then

$$\|I\|_{{}^cLM_{p_1\theta_1, \omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2, \omega_2}(\mathbb{R}^n, v_2)} \approx \sup_{x \in (0, \infty)} \varphi_1(x) \left(\int_0^\infty \mathcal{V}(t, x)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2, (t, \infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{1}{p_1 - \theta_2}}.$$

Theorem 2.10. Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$, $p_2 < p_1$ and $p_2 < \min\{\theta_1, \theta_2\}$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Suppose that \tilde{V} is admissible and

$$\varphi_2(x) := \left(\int_0^\infty [\tilde{V}(t) \mathcal{V}(x, t)]^{\theta_1 \rightarrow p_2} d\left(-\|\omega_1\|_{\theta_1, (0, t)}^{-\theta_1 \rightarrow p_2}\right) \right)^{\frac{1}{\theta_1 - p_2}} \in Q_{\tilde{V}^{-\frac{1}{p_1 - p_2}}}.$$

(i) If $\max\{p_1, \theta_1\} \leq \theta_2$, then

$$\begin{aligned} \|I\|_{{}^cLM_{p_1\theta_1, \omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2, \omega_2}(\mathbb{R}^n, v_2)} &\approx \sup_{x \in (0, \infty)} \varphi_2(x) \sup_{t \in (0, \infty)} \mathcal{V}(t, x) \|\omega_2\|_{\theta_2, (t, \infty)} \\ &\quad + \|\omega_1\|_{\theta_1, (0, \infty)}^{-1} \sup_{t \in (0, \infty)} \tilde{V}(t) \|\omega_2\|_{\theta_2, (t, \infty)}; \end{aligned}$$

(ii) If $p_1 \leq \theta_2 < \theta_1$, then

$$\begin{aligned} \|I\|_{{}^cLM_{p_1\theta_1, \omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2, \omega_2}(\mathbb{R}^n, v_2)} &\approx \left(\int_0^\infty \varphi_2(x)^{\frac{\theta_1 - \theta_2 - \theta_1 \rightarrow p_2}{\theta_2 \rightarrow p_2}} \tilde{V}(x)^{\theta_1 \rightarrow p_2} \left(\sup_{t \in (0, \infty)} \mathcal{V}(t, x) \|\omega_2\|_{\theta_2, (t, \infty)} \right)^{\theta_1 \rightarrow \theta_2} d\left(-\|\omega_1\|_{\theta_1, (0, x)}^{-\theta_1 \rightarrow p_2}\right) \right)^{\frac{1}{\theta_1 - \theta_2}} \\ &\quad + \|\omega_1\|_{\theta_1, (0, \infty)}^{-1} \sup_{t \in (0, \infty)} \tilde{V}(t) \|\omega_2\|_{\theta_2, (t, \infty)}; \end{aligned}$$

(iii) If $\theta_1 \leq \theta_2 < p_1$, then

$$\begin{aligned} \|I\|_{{}^cLM_{p_1\theta_1, \omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2, \omega_2}(\mathbb{R}^n, v_2)} &\approx \sup_{x \in (0, \infty)} \varphi_2(x) \left(\int_0^\infty \mathcal{V}(t, x)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2, (t, \infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{1}{p_1 - \theta_2}} \\ &\quad + \|\omega_1\|_{\theta_1, (0, \infty)}^{-1} \left(\int_0^\infty \tilde{V}(t)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2, (t, \infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{1}{p_1 - \theta_2}}; \end{aligned}$$

(iv) If $\theta_2 < \min\{p_1, \theta_1\}$, then

$$\begin{aligned} \|I\|_{{}^cLM_{p_1\theta_1, \omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2, \omega_2}(\mathbb{R}^n, v_2)} &\approx \left(\int_0^\infty \varphi_2(x)^{\frac{\theta_1 - \theta_2 - \theta_1 \rightarrow p_2}{\theta_2 \rightarrow p_2}} \tilde{V}(x)^{\theta_1 \rightarrow p_2} \left(\int_0^\infty \mathcal{V}(t, x)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2, (t, \infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{\theta_1 - \theta_2}{p_1 - \theta_2}} d\left(-\|\omega_1\|_{\theta_1, (0, x)}^{-\theta_1 \rightarrow p_2}\right) \right)^{\frac{1}{\theta_1 - \theta_2}} \\ &\quad + \|\omega_1\|_{\theta_1, (0, \infty)}^{-1} \left(\int_0^\infty \tilde{V}(t)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2, (t, \infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{1}{p_1 - \theta_2}}. \end{aligned}$$

Theorem 2.11. Let $0 < \theta_1 < p = p_1 = p_2 < \theta_2 < \infty$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$.

$$\|I\|_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t \in (0,\infty)} \left\| \|\omega_1\|_{\theta_1,(0,\cdot]}^{-1} \right\|_{\infty, v_1^{-1}v_2, B(0,t)} \|\omega_2\|_{\theta_2,(t,\infty)}.$$

Theorem 2.12. Let $0 < \theta_1, \theta_2 < \infty$ and $0 < p = p_1 = p_2 < \min\{\theta_1, \theta_2\}$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$ such that $v_1^{-1}v_2 \in C(\mathbb{R}^n)$. Suppose that $\omega_1 \in {}^c\Omega_{\theta_1}$, $\omega_2 \in \Omega_{\theta_2}$ and

$$0 < \|\omega_2^{-1}\|_{\theta_2 \rightarrow p, (x,\infty)} < \infty$$

holds for all $x > 0$.

(i) If $\theta_1 \leq \theta_2$, then

$$\begin{aligned} & \|I\|_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \\ & \approx \sup_{x \in (0,\infty)} \left(\widetilde{V}(x)^{\theta_1 \rightarrow p} \int_x^\infty d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p}\right) + \int_0^x \widetilde{V}(t)^{\theta_1 \rightarrow p} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p}\right) \right)^{\frac{1}{\theta_1 \rightarrow p}} \|\omega_2\|_{\theta_2,(x,\infty)} \\ & \quad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t \in (0,\infty)} \widetilde{V}(t) \|\omega_2\|_{\theta_2,(t,\infty)}; \end{aligned}$$

(ii) If $\theta_2 < \theta_1$, then

$$\begin{aligned} & \|I\|_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \\ & \approx \left(\int_0^\infty \left(\int_x^\infty d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p}\right) \right)^{\frac{\theta_1 \rightarrow \theta_2}{\theta_2 \rightarrow p}} \left(\sup_{0 < \tau \leq x} \widetilde{V}(\tau) \|\omega_2\|_{\theta_2,(\tau,\infty)} \right)^{\theta_1 \rightarrow \theta_2} d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \rightarrow p}\right) \right)^{\frac{1}{\theta_1 \rightarrow \theta_2}} \\ & \quad + \left(\int_0^\infty \left(\int_0^x \widetilde{V}(t)^{\theta_1 \rightarrow p} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p}\right) \right)^{\frac{\theta_1 \rightarrow \theta_2}{\theta_2 \rightarrow p}} \widetilde{V}(x)^{\theta_1 \rightarrow p} \|\omega_2\|_{\theta_2,(t,\infty)}^{\theta_1 \rightarrow \theta_2} d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \rightarrow p}\right) \right)^{\frac{1}{\theta_1 \rightarrow \theta_2}} \\ & \quad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t \in (0,\infty)} \widetilde{V}(t) \|\omega_2\|_{\theta_2,(t,\infty)}. \end{aligned}$$

3. PROOFS OF MAIN RESULTS

Before proceeding to the proof of our main results we recall the following integration in polar coordinates formula.

We denote the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ in \mathbb{R}^n by S^{n-1} . If $x \in \mathbb{R}^n \setminus \{0\}$, the polar coordinates of x are

$$r = |x| \in (0, \infty), \quad x' = \frac{x}{|x|} \in S^{n-1}.$$

There is a unique Borel measure $\sigma = \sigma_{n-1}$ on S^{n-1} such that if f is Borel measurable on \mathbb{R}^n and $f \geq 0$ or $f \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_{S^{n-1}} f(rx') r^{n-1} d\sigma(x') dr$$

(see, for instance, [12, p. 78]).

Lemma 3.1. Let $0 < p_1, p_2, \theta_1, \theta_2 \leq \infty$ and $p_1 < p_2$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Then ${}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \not\hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)$.

Proof. Assume that ${}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)$ holds. Then there exist $c > 0$ such that

$$\|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \leq c \|f\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}$$

holds for all $f \in \mathfrak{M}^+(\mathbb{R}^n)$. Let $\tau \in (0, \infty)$ and $f \in \mathfrak{M}(\mathbb{R}^n)$: $\text{supp } f \subset B(0, \tau)$. It is easy to see that

$$\begin{aligned} \|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} &= \left\| \|f\|_{p_2,v_2,B(0,t)} \right\|_{\theta_2,\omega_2,(0,\infty)} \\ &\geq \left\| \|f\|_{p_2,v_2,B(0,t)} \right\|_{\theta_2,\omega_2,(\tau,\infty)} \\ &\geq \|\omega_2\|_{\theta_2,(\tau,\infty)} \|f\|_{p_2,v_2,B(0,\tau)} \end{aligned} \tag{3.1}$$

and

$$\|f\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)} = \left\| \|f\|_{p_1,v_1,B(0,t)} \right\|_{\theta_1,\omega_1,(0,\infty)}$$

$$\begin{aligned}
(3.2) \quad &= \left\| \|f\|_{p_1, v_1, {}^cB(0, t)} \right\|_{\theta_1, \omega_1, (0, \tau)} \\
&\leq \|\omega_1\|_{\theta_1, (0, \tau)} \|f\|_{p_1, v_1, B(0, \tau)}.
\end{aligned}$$

Combining (3.1) with (3.2), we can assert that

$$\|\omega_2\|_{\theta_2, (\tau, \infty)} \|f\|_{p_2, v_2, B(0, \tau)} \leq c \|\omega_1\|_{\theta_1, (0, \tau)} \|f\|_{p_1, v_1, B(0, \tau)}.$$

Since $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$, we conclude that $L_{p_1}(B(0, \tau), v_1) \hookrightarrow L_{p_2}(B(0, \tau), v_2)$, which is a contradiction. \square

The following lemma is true.

Lemma 3.2. *Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$, $p_2 \leq p_1$ and $p_2 < \theta_2$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^c\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Then*

$$\|I\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)} = \left\{ \sup_{g \in \mathfrak{M}^+(0, \infty)} \frac{\|I\|^{p_2}_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \rightarrow L_{p_2}(v_2(\cdot)H^*g(\cdot|\cdot)^{\frac{1}{p_2}})}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2}, \omega_2^{-p_2}, (0, \infty)}} \right\}^{\frac{1}{p_2}}.$$

Proof. By duality, interchanging suprema, we have that

$$\begin{aligned}
&\|I\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)} \\
&= \sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \frac{\|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)}}{\|f\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1)}} \\
&= \sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \frac{1}{\|f\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1)}} \sup_{g \in \mathfrak{M}^+(0, \infty)} \frac{\left(\int_0^\infty \left(\int_{B(0, \tau)} f(x)^{p_2} v_2(x)^{p_2} dx \right) g(\tau) d\tau \right)^{\frac{1}{p_2}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2}, \omega_2^{-p_2}, (0, \infty)}^{\frac{1}{p_2}}} \\
&= \sup_{g \in \mathfrak{M}^+(0, \infty)} \frac{1}{\|g\|_{\frac{\theta_2}{\theta_2-p_2}, \omega_2^{-p_2}, (0, \infty)}^{\frac{1}{p_2}}} \sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{B(0, \tau)} f(x)^{p_2} v_2(x)^{p_2} dx \right) g(\tau) d\tau \right)^{\frac{1}{p_2}}}{\|f\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1)}}.
\end{aligned}$$

Applying Fubini's Theorem, we get that

$$\begin{aligned}
&\|I\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)} \\
&= \sup_{g \in \mathfrak{M}^+(0, \infty)} \frac{1}{\|g\|_{\frac{\theta_2}{\theta_2-p_2}, \omega_2^{-p_2}, (0, \infty)}^{\frac{1}{p_2}}} \sup_{f \in \mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} f(x)^{p_2} v_2(x)^{p_2} \left(\int_{|x|}^\infty g(\tau) d\tau \right) dx \right)^{\frac{1}{p_2}}}{\|f\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1)}} \\
(3.3) \quad &= \sup_{g \in \mathfrak{M}^+(0, \infty)} \frac{1}{\|g\|_{\frac{\theta_2}{\theta_2-p_2}, \omega_2^{-p_2}, (0, \infty)}^{\frac{1}{p_2}}} \|I\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \rightarrow L_{p_2}(v_2(\cdot)H^*g(\cdot|\cdot)^{\frac{1}{p_2}})}.
\end{aligned}$$

\square

Proof of Theorem 2.9. By Lemma 3.2, we have that

$$\|I\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2)} = \sup_{g \in \mathfrak{M}^+(0, \infty)} \frac{1}{\|g\|_{\frac{\theta_2}{\theta_2-p_2}, \omega_2^{-p_2}, (0, \infty)}^{\frac{1}{p_2}}} \|I\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \rightarrow L_{p_2}(v_2(\cdot)H^*g(\cdot|\cdot)^{\frac{1}{p_2}})}.$$

Since $\theta_1 \leq p_2$, applying [19, Theorem 4.2, (a)], we obtain that

$$\|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} \approx \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\sup_{t \in (0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p_2} \|H^*g(\cdot)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}}} \right\}^{\frac{1}{p_2}}.$$

Using polar coordinates, we have that

$$\|H^*g(\cdot)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)} = \|H^*g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,t)}, \quad t > 0,$$

where

$$\tilde{v}(r) := \int_{S^{n-1}} (v_1^{-1}v_2)(rx')^{\frac{p_1p_2}{p_1-p_2}} r^{n-1} d\sigma(x'), \quad r > 0.$$

Thus, we obtain that

$$\|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} \approx \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\sup_{t \in (0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p_2} \|H^*g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}}} \right\}^{\frac{1}{p_2}}.$$

Taking into account that

$$\begin{aligned} \int_0^t \tilde{v}(r) dr &= \int_0^t \int_{S^{n-1}} (v_1^{-1}v_2)(rx')^{\frac{p_1p_2}{p_1-p_2}} d\sigma(x') r^{n-1} dr \\ &= \int_{B(0,t)} (v_1^{-1}v_2)^{\frac{p_1p_2}{p_1-p_2}}(x) dx = \tilde{V}(t)^{\frac{p_1p_2}{p_1-p_2}}, \end{aligned} \quad (3.4)$$

(i) if $p_1 \leq \theta_2$, then applying [14, Theorem 3.2, (i)], we arrive at

$$\|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} \approx \sup_{x \in (0,\infty)} \varphi_1(x) \sup_{t \in (0,\infty)} \mathcal{V}(t,x) \|\omega_2\|_{\theta_2,(t,\infty)};$$

(ii) if $\theta_2 < p_1$, then applying [14, Theorem 3.2, (ii)], we arrive at

$$\|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} \approx \sup_{x \in (0,\infty)} \varphi_1(x) \left(\int_0^\infty \mathcal{V}(t,x)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{1}{p_1 \rightarrow \theta_2}}.$$

The proof is completed. □

Remark 3.3. In view of Remark 2.8, if

$$\begin{aligned} \limsup_{t \rightarrow 0+} \tilde{V}(t) \|\omega_1\|_{\theta_1,(0,t)}^{-1} &= \limsup_{t \rightarrow +\infty} \tilde{V}(t) \|\omega_1\|_{\theta_1,(0,t)} \\ &= \limsup_{t \rightarrow 0+} \|\omega_1\|_{\theta_1,(0,t)} = \limsup_{t \rightarrow +\infty} \|\omega_1\|_{\theta_1,(0,t)}^{-1} = 0, \end{aligned}$$

then $\varphi_1 \in Q_{\tilde{V}^{\frac{1}{p_1 \rightarrow p_2}}}$.

Proof of Theorem 2.10. By Lemma 3.2, applying [19, Theorem 4.2, (c)], we have that

$$\begin{aligned} \|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} &\approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\|H^*g(\cdot)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},\mathbb{R}^n}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}}} \right\}^{\frac{1}{p_2}} \\ &+ \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\left(\int_0^\infty \|H^*g(\cdot)\|_{\frac{\theta_1}{\theta_1-p_2},\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1 p_2}{\theta_1-p_2}}\right) \right)^{\frac{\theta_1-p_2}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}}} \right\}^{\frac{1}{p_2}}. \end{aligned}$$

Using polar coordinates, we have that

$$\begin{aligned} \|I\|_{{}^cLM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} &\approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{\frac{p_1}{p_1-p_2}, \tilde{v}, \frac{p_1-p_2}{p_1}, (0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2}, \omega_2^{-p_2}, (0,\infty)}} \right\}^{\frac{1}{p_2}} \\ &+ \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\left(\int_0^\infty \|H^*g\|_{\frac{\theta_1}{\theta_1-p_2}, \frac{p_1}{p_1-p_2}, \tilde{v}, (0,t)}^{\frac{\theta_1-p_2}{p_1}} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1-p_2}{p_1}}\right) \right)^{\frac{\theta_1-p_2}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2}, \omega_2^{-p_2}, (0,\infty)}} \right\}^{\frac{1}{p_2}} \\ &:= C_1 + C_2. \end{aligned}$$

Assume first that $p_1 \leq \theta_2$. On using the characterization of the boundedness of the operator H^* in weighted Lebesgue spaces (see, for instance, [17, 20]), we arrive at

$$C_1 \approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t \in (0,\infty)} \tilde{V}(t) \|\omega_2\|_{\theta_2,(t,\infty)}.$$

(i) Let $\theta_1 \leq \theta_2$. Applying [14, Theorem 3.1, (i)], we obtain that

$$C_2 \approx \sup_{x \in (0,\infty)} \varphi_2(x) \sup_{t \in (0,\infty)} \mathcal{V}(t, x) \|\omega_2\|_{\theta_2,(t,\infty)}.$$

Consequently, the proof is completed in this case.

(ii) Let $\theta_2 < \theta_1$. Using [14, Theorem 3.1, (ii)], we have that

$$C_2 \approx \left(\int_0^\infty \varphi_2(x)^{\frac{\theta_1 \rightarrow \theta_2, \theta_1 \rightarrow p_2}{\theta_2 \rightarrow p_2}} \tilde{V}(x)^{\theta_1 \rightarrow p_2} \left(\sup_{t \in (0,\infty)} \mathcal{V}(t, x) \|\omega_2\|_{\theta_2,(t,\infty)} \right)^{\theta_1 \rightarrow \theta_2} d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \rightarrow p_2}\right) \right)^{\frac{1}{\theta_1 \rightarrow \theta_2}},$$

and the statement follows in this case.

Let us now assume that $\theta_2 < p_1$. Then, using the characterization of the boundedness of the operator H^* in weighted Lebesgue spaces, we have that

$$C_1 \approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left(\int_0^\infty \tilde{V}(t)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{1}{p_1 \rightarrow \theta_2}}.$$

(iii) Let $\theta_1 \leq \theta_2$, then [14, Theorem 3.1, (iii)] yields that

$$C_2 \approx \sup_{x \in (0,\infty)} \varphi_2(x) \left(\int_0^\infty \mathcal{V}(t, x)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{1}{p_1 \rightarrow \theta_2}},$$

and these completes the proof in this case.

(iv) If $\theta_2 < \theta_1$, then on using [14, Theorem 3.1, (iv)], we arrive at

$$C_2 \approx \left(\int_0^\infty \varphi_2(x)^{\frac{\theta_1 \rightarrow \theta_2, \theta_1 \rightarrow p_2}{\theta_2 \rightarrow p_2}} \tilde{V}(x)^{\theta_1 \rightarrow p_2} \left(\int_0^\infty \mathcal{V}(t, x)^{p_1 \rightarrow \theta_2} d\left(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \rightarrow \theta_2}\right) \right)^{\frac{\theta_1 \rightarrow \theta_2}{p_1 \rightarrow \theta_2}} d\left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \rightarrow p_2}\right) \right)^{\frac{1}{\theta_1 \rightarrow \theta_2}},$$

and in this case the proof is completed. \square

Remark 3.4. Assume that $\varphi_2(x) < \infty$, $x > 0$. In view of Remark 2.7, if

$$\int_0^1 \left(\int_0^t \omega_1^{\theta_1} \right)^{-\frac{\theta_1}{\theta_1-p_2}} \omega_1^{\theta_1}(t) dt = \int_1^\infty \tilde{V}(t)^{\frac{\theta_1 p_2}{\theta_1-p_2}} \left(\int_0^t \omega_1^{\theta_1} \right)^{-\frac{\theta_1}{\theta_1-p_2}} \omega_1^{\theta_1}(t) dt = \infty,$$

then $\varphi_2 \in \mathcal{Q}_{\tilde{V}^{\frac{1}{p_1 \rightarrow p_2}}}$.

Proof of Theorem 2.11. By Lemma 3.2, applying [19, Theorem 4.2, (b)], we get that

$$\|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} = \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\sup_{t \in (0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p} \|H^*g(|\cdot|)\|_{\infty,(v_1^{-1}v_2)^p,B(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}.$$

Recall that, whenever F, G are non-negative measurable functions on $(0, \infty)$ and F is non-increasing, then

$$(3.5) \quad \operatorname{ess\,sup}_{t \in (0,\infty)} F(t)G(t) = \operatorname{ess\,sup}_{t \in (0,\infty)} F(t) \operatorname{ess\,sup}_{\tau \in (0,t)} G(\tau).$$

Observe that

$$(3.6) \quad \|H^*g(|\cdot|)\|_{\infty,(v_1^{-1}v_2)^p,B(0,t)} = \sup_{\tau \in (0,t)} \sup_{|y|=\tau} (v_1^{-1}(y)v_2(y))^p H^*g(|y|) = \|H^*g\|_{\infty,\tilde{v},(0,t)}$$

holds for all $t > 0$, where $\tilde{v}(\tau) := (\sup_{|y|=\tau} v_1^{-1}(y)v_2(y))^p$, $\tau > 0$.

On using (3.5), we get that

$$\begin{aligned} \|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} &= \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\sup_{t \in (0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p} \|H^*g\|_{\infty,\tilde{v},(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}} \\ &= \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{\infty,\|\omega_1\|_{\theta_1,(0,\cdot)}^{-p},\tilde{v}(\cdot),(0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}. \end{aligned}$$

Using the characterization of the boundedness of H^* in weighted Lebesgue spaces, we obtain that

$$\begin{aligned} \|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} &\approx \sup_{t \in (0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{s \in (0,t)} \|\omega_1\|_{\theta_1,(0,s)}^{-1} \tilde{v}(s)^{\frac{1}{p}} \right) \\ &= \sup_{t \in (0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{s \in (0,t)} \sup_{|y|=s} \|\omega_1\|_{\theta_1,(0,|y|)}^{-1} v_1^{-1}(y)v_2(y) \right) \\ &= \sup_{t \in (0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{x \in B(0,t)} \|\omega_1\|_{\theta_1,(0,|x|)}^{-1} v_1^{-1}(x)v_2(x) \right) \\ &= \sup_{t \in (0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left\| \|\omega_1\|_{\theta_1,(0,|\cdot|)}^{-1} \right\|_{\infty,v_1^{-1}v_2,B(0,t)}. \end{aligned}$$

□

Proof of Theorem 2.12. By Lemma 3.2, applying [19, Theorem 4.2, (d)], and using (3.6), we get that

$$\begin{aligned} \|I\|_{c_{LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \rightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}} &\approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{\infty,\tilde{v},(0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}} \\ &\quad + \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\left(\int_0^\infty \|H^*g\|_{\infty,\tilde{v},(0,t)}^{\frac{\theta_1}{\theta_1-p}} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1 p}{\theta_1-p}}\right) \right)^{\frac{\theta_1-p}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}} \\ &:= C_3 + C_4. \end{aligned}$$

Again, using the characterization of the boundedness of H^* in weighted Lebesgue spaces, we obtain that

$$C_3 \approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t \in (0,\infty)} \tilde{V}(t) \|\omega_2\|_{\theta_2,(t,\infty)}.$$

(i) Let $\theta_1 \leq \theta_2$, then by [15, Theorem 4.1], we have that

$$C_4 \approx \sup_{x \in (0,\infty)} \left(\tilde{V}(x)^{\theta_1 \rightarrow p} \int_x^\infty d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p}\right) + \int_0^x \tilde{V}(t)^{\theta_1 \rightarrow p} d\left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p}\right) \right)^{\frac{1}{\theta_1 \rightarrow p}} \|\omega_2\|_{\theta_2,(x,\infty)},$$

and the statement follows in this case.

(ii) Let $\theta_2 < \theta_1$, then [15, Theorem 4.4] yields that

$$C_4 \approx \left(\int_0^\infty \left(\int_x^\infty d \left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p} \right) \right)^{\frac{\theta_1 \rightarrow \theta_2}{\theta_2 \rightarrow p}} \left(\sup_{0 < \tau \leq x} \widetilde{V}(\tau) \|\omega_2\|_{\theta_2,(\tau,\infty)} \right)^{\theta_1 \rightarrow \theta_2} d \left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \rightarrow p} \right) \right)^{\frac{1}{\theta_1 \rightarrow \theta_2}} \\ + \left(\int_0^\infty \left(\int_0^x \widetilde{V}(t)^{\theta_1 \rightarrow p} d \left(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \rightarrow p} \right) \right)^{\frac{\theta_1 \rightarrow \theta_2}{\theta_2 \rightarrow p}} \widetilde{V}(x)^{\theta_1 \rightarrow p} \|\omega_2\|_{\theta_2,(t,\infty)}^{\theta_1 \rightarrow \theta_2} d \left(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \rightarrow p} \right) \right)^{\frac{1}{\theta_1 \rightarrow \theta_2}},$$

and the proof is completed in this case. \square

Acknowledgments. The research of A. Gogatishvili was partially supported by the grant P201-13-14743S of the Grant Agency of the Czech Republic and RVO: 67985840 and by Shota Rustaveli National Science Foundation grants no. DI/9/5-100/13 (Function spaces, weighted inequalities for integral operators and problems of summability of Fourier series).

REFERENCES

- [1] Ts. Batbold and Y. Sawano, *Decompositions for local Morrey spaces*, Eurasian Math. J. **5** (2014), no. 3, 9–45.
- [2] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. I*, Eurasian Math. J. **3** (2012), no. 3, 11–32.
- [3] V. I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces. II*, Eurasian Math. J. **4** (2013), no. 1, 21–45.
- [4] V. I. Burenkov and M. L. Goldman, *Necessary and sufficient conditions for the boundedness of the maximal operator from Lebesgue spaces to Morrey-type spaces*, Math. Inequal. Appl. **17** (2014), no. 2, 401–418, DOI 10.7153/mia-17-30.
- [5] V. I. Burenkov and H. V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces*, Studia Math. **163** (2004), no. 2, 157–176.
- [6] V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, *Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces*, J. Comput. Appl. Math. **208** (2007), no. 1, 280–301.
- [7] V.I. Burenkov, H.V. Guliyev, and V.S. Guliyev, *On boundedness of the fractional maximal operator from complementary Morrey-type spaces to Morrey-type spaces*, The interaction of analysis and geometry, Contemp. Math., vol. 424, Amer. Math. Soc., Providence, RI, 2007, pp. 17–32.
- [8] V.I. Burenkov, V.S. Guliyev, A. Serbetci, and T.V. Tararykova, *Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces*, Eurasian Math. J. **1** (2010), no. 1, 32–53.
- [9] V.I. Burenkov, A. Gogatishvili, V.S. Guliyev, and R.Ch. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. **55** (2010), no. 8-10, 739–758.
- [10] V.I. Burenkov, A. Gogatishvili, V.S. Guliyev, and R.Ch. Mustafayev, *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **35** (2011), no. 1, 67–87.
- [11] V.I. Burenkov and E.D. Nursultanov, *Description of interpolation spaces for local Morrey-type spaces*, Tr. Mat. Inst. Steklova **269** (2010), no. Teoriya Funktsii i Differentsialnye Uravneniya, 52–62 (Russian, with Russian summary); English transl., Proc. Steklov Inst. Math. **269** (2010), no. 1, 46–56.
- [12] G. B. Folland, *Real analysis*, 2nd ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999. Modern techniques and their applications; A Wiley-Interscience Publication.
- [13] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, 1983.
- [14] A. Gogatishvili, R. Ch. Mustafayev, and L.-E. Persson, *Some new iterated Hardy-type inequalities*, J. Funct. Spaces Appl. (2012), Art. ID 734194, 30.
- [15] A. Gogatishvili, B. Opic, and L. Pick, *Weighted inequalities for Hardy-type operators involving suprema*, Collect. Math. **57** (2006), no. 3, 227–255.
- [16] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in \mathbb{R}^n* , Doctor's degree dissertation. Mat. Inst. Steklov., Moscow, 1994 (Russian).
- [17] A. Kufner and L.-E. Persson, *Weighted inequalities of Hardy type*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003. MR1982932 (2004c:42034)
- [18] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), no. 1, 126–166, DOI 10.2307/1989904.
- [19] R. Ch. Mustafayev and T. Ünver, *Embeddings between weighted local Morrey-type spaces and weighted Lebesgue spaces*, J. Math. Inequal. **9** (2015), no. 1, 277–296, DOI 10.7153/jmi-09-24.
- [20] B. Opic and A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Harlow, 1990.

INSTITUTE OF MATHEMATICS, ACADEMY OF SCIENCES OF THE CZECH REPUBLIC, ŽITNÁ 25, 115 67 PRAHA 1, CZECH REPUBLIC
E-mail address: gogatish@math.cas.cz

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, KIRIKKALE UNIVERSITY, 71450 YAHSIHAN, KIRIKKALE, TURKEY
E-mail address: rzamustafayev@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, KIRIKKALE UNIVERSITY, 71450 YAHSIHAN, KIRIKKALE, TURKEY
E-mail address: tugceunver@gmail.com